Black Hole Thermodynamics from Entanglement Mechanics

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Black Hole Thermodynamics and Problem of Universality

- For small perturbations to a stationary black hole:

$$dM = \frac{\kappa}{8\pi} dA + \text{work terms} \quad \Leftrightarrow \quad dE = TdS + \text{work terms} \qquad (\textit{first law})$$

- The integral form of first law is Komar relation, or equivalently, the generalized Smarr formula:

$$E_{Komar} = 2T_H S_{BH} \quad \Leftrightarrow \quad M = 2T_H S_{\rm BH} + 2\Omega_H J + \Phi_H Q + \ldots$$

- Entropy S_{BH} and Hawking temperature T_H are of quantum origin ($k_B = c = G = 1$):

$$S_{BH} = \frac{A}{4\hbar} \quad ; \quad T_H = \frac{\hbar\kappa}{2\pi}$$

Black Hole Thermodynamics and Problem of Universality

BH entropy can be derived by assuming various degrees of freedom:

- Noether Charge [*Wald '93*]
- D-Branes [*Strominger & Vafa '96*]
- Spin-networks [*Ashtekar et al '96*]
- Conformal Symmetry [*Carlip '99*]
- Entanglement Entropy [*Bombelli et al '86, Das & SS '08*]

How can we break this degeneracy? ("Universality Problem") [*Carlip '07*]

Without assuming any fundamental structure, entanglement entropy of free fields is :

$$S \sim c_0 \frac{A}{a^2} \quad \Leftrightarrow \quad S_{BH} = \frac{A}{4}$$

How to quantify entanglement of a quantum system?

Hamiltonian for a coupled harmonic oscillator ($\hbar = m = 1$):

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega^2}{2}\left(x_1^2 + x_2^2\right) + \frac{\alpha^2}{2}\left(x_1 - x_2\right)^2 \;\Leftrightarrow\; \frac{p_+^2}{2} + \frac{p_-^2}{2} + \frac{1}{2}\omega_+^2 x_+^2 + \frac{1}{2}\omega_-^2 x_-^2 \tag{1}$$

where $\omega_- = \sqrt{\omega^2 + 2\alpha^2}$ and $\omega_+ = \omega$. The GS wavefunction is entangled :

$$\Psi_0(x_+, x_-) = \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}} \exp\left\{-\frac{\omega_+ x_+^2}{2} - \frac{\omega_- x_-^2}{2}\right\} \neq \Psi(x_1)\Psi(x_2) \tag{2}$$

The entanglement entropy of the system is defined as:

$$S = -\,\mathrm{Tr}\,\rho_1 \ln \rho_1 \quad ; \quad \rho_1 = Tr_2 |\Psi_0\rangle\langle\Psi_0| \tag{3}$$

How to quantify entanglement of a quantum system?

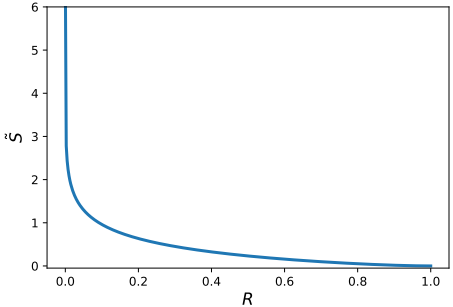


Figure: Entanglement entropy of CHO w.r.t coupling parameter $R = \omega_+/\omega_-$.

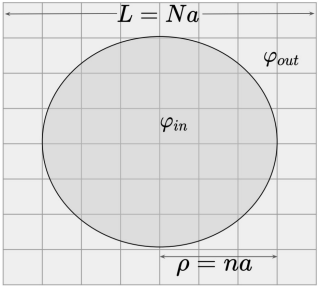
Important limits :

- $R = 0 \implies \omega = 0 || \alpha = \infty$: Entropy diverges
- $R = 1 \implies \omega = \infty || \alpha = 0$: Entropy vanishes

Spherically Symmetric Space-times with Horizon(s)

Consider a line element with a horizon r_h s.t. $f(r_h) = 0$:

$$ds^2 = -f(r)dt^2 + d\rho^2 + r^2 d\Omega^2 \quad ; \quad \rho = \int_{r_h}^{r} \frac{dr}{\sqrt{f(r)}}$$



Action of massless scalar field:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

After partial wave expansion & lattice-regularization ($\rho = ja$):

$$H = \frac{1}{2a} \sum_{lmj} \left[\pi^2_{lm,j} + r^2_{j+\frac{1}{2}} f^{1/2}_{j+\frac{1}{2}} \left\{ f^{1/4}_j \frac{\varphi_{lm,j}}{r_j} - f^{1/4}_{j+1} \frac{\varphi_{lm,j+1}}{r_{j+1}} \right\}^2 + \frac{l(l+1)}{r_j^2} f_j \varphi^2_{lm,j} \right] = \frac{1}{a} \sum_{lm} H_{lm} \qquad (4)$$

Entanglement Entropy

The Hamiltonian can be bipartited as follows:

$$H_{lm} = \frac{1}{2}\left[\sum_i \pi^2_{lm,i} + \sum_{ij} K_{ij}\varphi_{lm,i}\varphi_{lm,j}\right] = \begin{bmatrix} H_{in} & H_{int} \\ H_{int} & H_{out} \end{bmatrix}. \tag{5}$$

GS entanglement mechanics of φ_{in}:

- Entanglement entropy :

$$S_{lm} = -Tr\rho_{red}\ln\rho_{red} \quad ; \quad S = \sum_l (2l+1)S_{lm} \quad ; \quad \rho_{red} = \mathrm{Tr}_{1,..,n}|\Psi_0\rangle\langle\Psi_0| \tag{6}$$

- Entanglement energy

[*Mukohyama et al '98, SMC & SS '20*]

$$E = \epsilon\int\prod_{A=1}^{N} d\bar{\varphi}_{lm,A}\,\langle\{\bar{\varphi}_{lm,B}\}|:H_{in}:\rho|\{\bar{\varphi}_{lm,C}\}\rangle \tag{7}$$

Schwarzschild Black Hole

$$f(r) = 1 - 2M/r \quad ; \quad \Delta_M = M/a$$

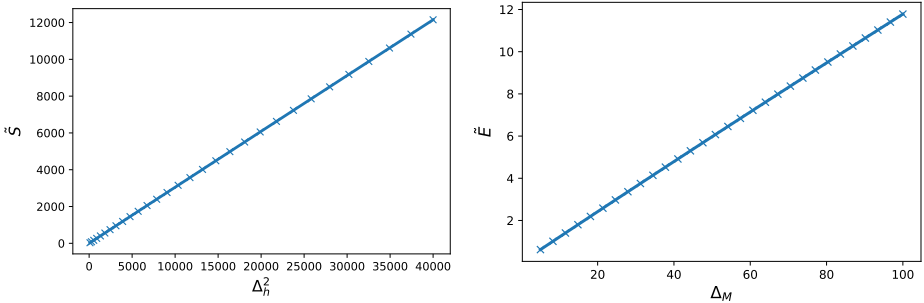


Figure: Entanglement Mechanics for Schwarzschild Black Hole.

$$S = c_s \frac{r_h^2}{a^2}; \quad E = c_e \frac{M}{a^2}; \quad \frac{E}{2S} = \frac{\pi c_e}{c_s} T_H$$

Reissner-Nordström Event Horizon

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad ; \quad \Delta_M = M/a; \quad \Delta_Q = Q/a; \quad \Delta_+ = r_+/a$$

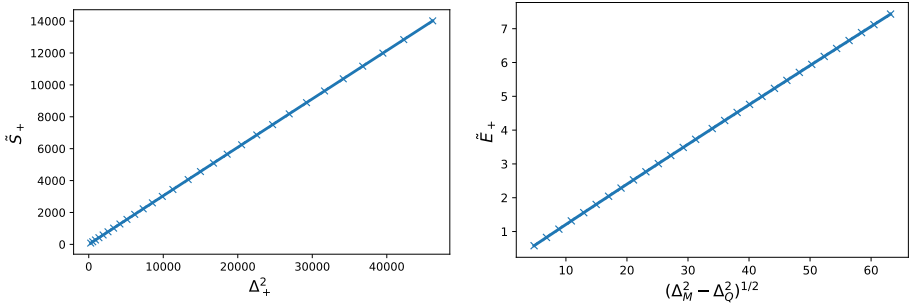


Figure: Entanglement Mechanics at RN event horizon, when $M/Q = 1.1$.

$$S_+ = c_s \frac{r_+^2}{a^2}; \quad E_+ = c_e \frac{\sqrt{M^2 - Q^2}}{a^2}; \quad \frac{E_+}{2S_+} = \frac{\pi c_e}{c_s} T_H^{(+)}$$

One-to-One Correspondence

From simulations for Schwarzschild, de Sitter, RNBH, Schwarzschild-AdS and Schwarzschild-dS space-times:

- Proportionality constants are universal for all black holes studied:

$$c_e \sim 0.0955; \quad c_s \sim 0.3$$

- We observe a one-to-one correspondence as follows:

$$E = \frac{c_e}{a^2} E_{Komar}; \quad S = \frac{c_s}{\pi a^2} S_{BH}$$

- We obtain a universal relation similar to the Komar relation from BH thermodynamics:

$$E = 2T_H S \quad \Leftrightarrow \quad E_{Komar} = 2T_H S_{BH}$$

Smarr-Formula from Entanglement Scaling Relations

Space-time	Entanglement Mechanics	Thermodynamic Structure	Smarr formula	Pressure	Potential
Schwarzschild	$S = (c_s/a^2)r_h^2$ $E = (c_e/a^2)M$	$S_{BH} = \pi r_h^2$ $E_{Komar} = M$	$M = 2T_H S_{BH}$	—	—
Reissner-Nordström	$S_+ = (c_s/a^2)r_+^2$ $E_+ = (c_e/a^2)\sqrt{M^2 - Q^2}$	$S_{BH} = \pi r_+^2$ $E_{Komar} = \sqrt{M^2 - Q^2}$	$M = 2T_H S_{BH} + Q^2/r_+$	—	Q/r_+
Schwarzschild-AdS	$S = (c_s/a^2)r_h^2$ $E = (c_e/a^2)[3M - r_h^2]$	$S_{BH} = \pi r_h^2$ $E_{Komar} = 3M - r_h^2$	$M = 2T_H S_{BH} - r_h^3/l^2$	$3/8\pi l^2$	—
Schwarzschild-dS	$S_b = (c_s/a^2)r_b^2$ $E_b = (c_e/a^2)[3M - r_b^2]$	$S_{BH} = \pi r_b^2$ $E_{Komar} = 3M - r_b^2$	$M = 2T_H S_{BH} + r_b^3/l^2$	$-3/8\pi l^2$	—

Table: Summary of entanglement mechanics and event-horizon thermodynamics. They also satisfy $E = 2T_H S$ universally.

Conclusions and Future Directions

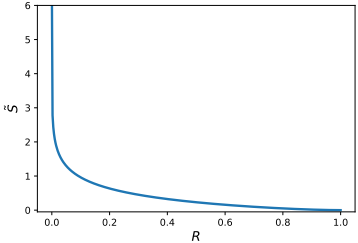
- One-to-one correspondence between entanglement mechanics (E, S) and black-hole thermodynamics (E_{Komar}, S_{BH}).
- Entanglement mechanics do not explicitly use Einstein's equations. However, the scalar field φ captures information about the dynamical properties of background space-time!
- Komar relation and Smarr formula are exactly recovered. New, independent derivation of integral form of first law from entanglement.
- Does this hold in Kerr/modified theories/higher dimensions?

Thank You!

How to quantify entanglement of a quantum system?

The reduced density matrix of $\{x_1\}$ oscillator is:

$$\rho_1(x_1, x_1') = \int_{-\infty}^{\infty} dx_2 \Psi_0^*(x_1', x_2)\Psi_0(x_1, x_2)$$



The entanglement entropy of the system is therefore:

$$S = -\operatorname{Tr}\rho_1 \ln \rho_1 = -\ln[1-\xi] - \frac{\xi}{1-\xi}\ln\xi$$

where

$$\xi = \left[\frac{(\lambda+2)^{1/4} - \lambda^{1/4}}{(\lambda+2)^{1/4} + \lambda^{1/4}}\right]^2 \quad ; \quad \lambda = \frac{\omega^2}{\alpha^2} = \frac{4R^2}{1-R^2}$$

Entanglement Energy

We define entanglement energy as follows: [Mukohyama et al '98, SMC & SS '20]

$$E = \epsilon \int \prod_{A=1}^{N} d\bar{\varphi}_{lm,A} \langle \{\bar{\varphi}_{lm,B}\} | : H_{in} : \rho | \{\bar{\varphi}_{lm,C}\} \rangle \,, \tag{8}$$

where $\omega^{in} = K_{in}^{1/2}$ and the normal ordered Hamiltonian H_{in} is given by:

$$\begin{aligned} : H_{in} : &= -\frac{1}{2}\delta^{ab}\left(\frac{\partial}{\partial\varphi^{a}_{lm}} - \omega^{in}_{ac}\varphi^{c}_{lm}\right)\left(\frac{\partial}{\partial\varphi^{b}_{lm}} + \omega^{in}_{bd}\varphi^{d}_{lm}\right) \\ &= -\frac{1}{2}U^{ab}\left(\frac{\partial}{\partial\bar{\varphi}^{A}_{lm}} - \bar{\omega}^{in}_{AC}\bar{\varphi}^{C}_{lm}\right)\left(\frac{\partial}{\partial\bar{\varphi}^{B}_{lm}} + \bar{\omega}^{in}_{BD}\bar{\varphi}^{D}_{lm}\right). \end{aligned} \tag{9}$$

Schwarzschild Black Hole

Can be described by the following line element and proper length:

$$f(r) = 1 - 2M/r \qquad ; \qquad \rho = r\sqrt{1 - \frac{2M}{r}} + M \ln\left[\frac{r}{2M}\left\{1 + \sqrt{1 - \frac{2M}{r}}\right\}^2\right] \tag{10}$$

On discretizing $\rho = ja$, we get a scale-invariant expression that connects lattice-points in the proper length and radial co-ordinates as follows:

$$j = r_j\sqrt{1 - \frac{2\Delta_M}{r_j}} + \Delta_M \ln\left[\frac{r_j}{2\Delta_M}\left\{1 + \sqrt{1 - \frac{2\Delta_M}{r_j}}\right\}^2\right] \quad ; \quad f_j = 1 - 2\Delta_M/r_j \tag{11}$$

where $\Delta_M = M/a$ and $r_j = r/a$ are dimensionless.

Reissner-Nordström Event Horizon

Can be described by the following line element:

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{12}$$

For $Q < M$, the roots are given by $r_\pm = M \pm \sqrt{M^2 - Q^2}$. Thus, $f(r)$ is positive in two regions: $0 < r < r_-$ and $r_+ < r < \infty$.

$$j = \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)} + \chi\Delta_Q \ln\left[\frac{r_j - \chi\Delta_Q + \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)}}{\Delta_Q\sqrt{\chi^2 - 1}}\right] \tag{13}$$

where $\Delta_Q \equiv Q/a$, $\chi = M/Q$ and $r_j \equiv r/a$ are all dimensionless

SdS Cosmological Horizon

Can be described by the following line element:

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{l^2} \tag{14}$$

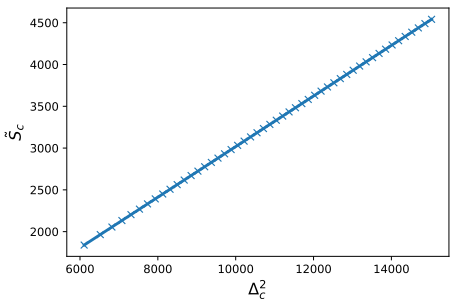
Once more, we have two horizons — r_b (event horizon) and r_c (cosmological horizon):

$$r_b = \frac{2l}{\sqrt{3}} \cos \frac{\pi + \theta}{3}; \quad r_c = \frac{2l}{\sqrt{3}} \cos \frac{\pi - \theta}{3} \tag{15}$$

where $\theta = \cos^{-1}(3\sqrt{3}\chi)$ and $\chi = M/l \in \left[0, 1/(3\sqrt{3})\right]$. $f(r)$ is positive in the region between the two horizons.

SdS Cosmological Horizon

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{l^2} \quad ; \quad \Delta_M = M/a; \quad \Delta_l = l/a; \quad \Delta_c = r_c/a$$



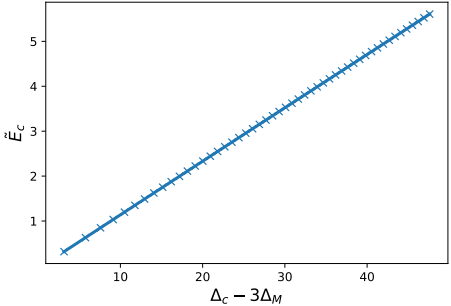


Figure: Entanglement Mechanics at SdS cosmological horizon, when $\Delta_M = 25$.

$$S_{c,N} \sim c_s \frac{r_{c,N}^2}{a^2}; \quad E_{c,N} \sim c_e \frac{r_{c,N} - 3M}{a^2}; \quad \frac{E_c}{2S_c} = \frac{\pi c_e}{c_s} T_H^{(c)}$$