Black Hole Thermodynamics from Entanglement Mechanics

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Since its inception, the Bekenstein-Hawking area relation for black-hole entropy has been the primary testing ground for various theories of quantum gravity. However, a key challenge to such theories is identifying the microscopic structures and explaining the exponential growth of microstates, providing a fundamental understanding of thermodynamic quantities. Since entropy is a single number, we explore other quantities to provide complete information about the black-hole microstates. We establish a one-to-one correspondence between entanglement energy, entropy, and temperature (quantum entanglement mechanics) and the Komar energy, Bekenstein-Hawking entropy, and Hawking temperature of the horizon (black-hole thermodynamics), respectively. We also show that this correspondence leads to the Komar relation and Smarr formula for generic 4-D spherically symmetric space-times. While offering an independent derivation of black-hole thermodynamics from field observables, the universality of results suggests that quantum entanglement is a *fundamental building block of space-time*.

Keywords: Entanglement Entropy; Black Hole Thermodynamics; Field Theory in Curved Space-times.

1. Introduction

Black holes are fascinating entities that typically arise from gravitationally collapsing bodies, such as stars at the end of their life cycle or star collisions. Despite their violent origin, black holes relax to a stationary state, which can then be fully described by a mere handful of variables such as their mass, charge and angular momentum. This makes it easier to probe various quantum and gravitational phenomena that come with it, further making it an ideal testing ground for ongoing research on quantum gravity.

Interestingly, like ordinary matter systems such as ideal gases, black holes also obey an equation of state. The physical parameters describing stationary black-holes satisfy what is known as the Komar relation[1–3]:

$$E_{\rm Komar} = 2\, T_{\rm H}\, S_{\rm BH}, \tag{1}$$

where $E_{\rm Komar}$ is related to the Hamiltonian of the Einstein-Hilbert action[4], $T_{\rm H}$ is the Hawking temperature[5] and $S_{\rm BH}$ is the Bekenstein-Hawking entropy[6]. The above relation is a by-product of the Smarr formula[7,8]:

$$M = 2T_H\, S_{\rm BH} + 2\Omega_H J + \Phi_H Q \tag{2}$$

that relates the mass (M), angular momentum (J), entropy ($S_{\rm BH}$), electric charge (Q) of black-holes, angular velocity Ω_H, and Φ_H which is the potential difference between the horizon and infinity. It was shown that the Smarr formula, in its differential form and subject to certain assumptions, gives rise to the first law of black hole thermodynamics [9]. At the heart of this law is the analogy that connects thermodynamic entropy to surface area, and temperature to surface gravity.

Most of the effort in literature has been to understand the microscopic statistical origin of black-hole entropy [10–13], and in extension, black hole thermodynamics. However, black-hole entropy has the problem of Universality [14]— wherein up to the leading order, several approaches using completely different microscopic degrees of freedom lead to Bekenstein-Hawking entropy [10,13,14]. The inability to conclusively distinguish such approaches, along with the fact that entropy is a single number, suggests that it may be impossible to identify the true degrees of freedom that give rise to black hole entropy. Therefore, it requires us to also identify other physical quantities corresponding to black hole thermodynamics, if we were to attempt resolving the universality problem.

Over the last decade or so, one such approach that has grown increasingly crucial to understanding black-hole physics, and quantum gravity in general, is quantum entanglement [15–19]. A fundamental feature of entanglement is the *Area law* — the entanglement entropy of blocks of low energy states of local Hamiltonian is often proportional to the measure of the boundary separating the block from its setting [13,20–22]. The area law has not only established a direct link between entanglement entropy and black hole entropy, but also led to the conjecture that space-time fabric might itself be built upon entanglement [17]. If entanglement is a necessity for the existence of space-time, then the *litmus-test* is to derive all the quantities of black-hole thermodynamics from entanglement.

In this talk, we show that the quantum scalar fields in a $(3 + 1)$-dimensional black-hole space-times with one or more horizons, provide a way to obtain the physical quantities from entanglement *aka entanglement mechanics*. We relate these quantities to black-hole thermodynamics, and further show that these quantities satisfy Smarr formula (2) [23].

2. Modeling scalar field in a background space-time

In this section, we recapitulate the procedure for probing entanglement mechanics of a minimally coupled scalar field against a static, spherically symmetric background space-time with one or more horizons. The line-element for such a space-time can in general be written as follows:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2. \tag{3}$$

Depending on the form of $f(r)$, there is usually a handful of coordinate settings that help us understand the system better. Suppose the space-time in question has a horizon (r_h), such as in the Schwarzschild case, it is useful to rewrite the metric in terms of proper-length coordinates[24]:

$$ds^2 = -f(r)dt^2 + d\rho^2 + r^2 d\Omega^2 \qquad ; \qquad \rho = \int_{r_h}^{r} \frac{dr}{\sqrt{f(r)}}. \tag{4}$$

Before modeling the scalar field in this setting, we have to address two fundamental aspects — i) Since entanglement entropy diverges in the continuum limit, the field has to be *regularized*. The regularisation provided by the more rigorous definition of quantum fields in terms of operator-valued distributions is difficult to calculate even for simple free theories[25]. We therefore resort to regularisation by discretization[21,22], i.e., we treat quantum fields on a lattice, with the lattice spacing a fixed in units of proper length as depicted in Fig 1. (ii) To capture entanglement of the field across a space-time horizon, the field must be treated as a *bipartite* system made up of subsystems on either sides of the horizon. However, the proper length co-ordinates only capture the region on that side of the horizon where $f(r) > 0$. The physics in the region $f(r) < 0$ will hence remain inaccessible. The bipartition will then have to be performed as close to the horizon as possible, i.e., by tracing out a single degree of freedom along proper length co-ordinate. This is in fact a valid approximation for large horizon radius and/or small lattice spacing of discretized field[23].

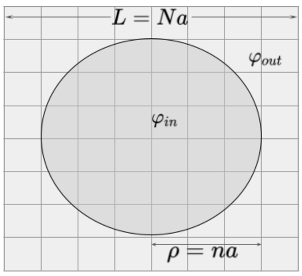


Fig. 1. Discretization scheme for a scalar field in proper length co-ordinates. The field essentially is replaced by a network of coupled harmonic oscillators placed at every grid point.

To set the model up, we begin with the action for a massive scalar field in an arbitrary space-time[26,27]:

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m_f^2\varphi^2\right]. \tag{5}$$

For the proper-length coordinate (4), we use the following spherical decomposition of the scalar field with appropriate scaling:

$$\dot{\varphi}(\rho,\theta,\phi) = \frac{f^{1/4}(r)}{r} \sum_{lm} \dot{\varphi}_{lm}(\rho) Z_{lm}(\theta,\phi) \tag{6}$$

$$\varphi(\rho,\theta,\phi) = \frac{f^{1/4}(r)}{r} \sum_{lm} \varphi_{lm}(\rho) Z_{lm}(\theta,\phi). \tag{7}$$

Substituting these in the action (5), leads to the following effective $(1+1)$-D Lagrangian:

$$L = \frac{1}{2}\sum_{lm}\int d\rho \left[\dot{\varphi}_{lm}^2 - r^2\sqrt{f(r)}\left\{\partial_\rho\left(f^{1/4}(r)\frac{\varphi_{lm}}{r}\right)\right\}^2 - f(r)\left\{m_f^2 + \frac{l(l+1)}{r^2}\right\}\varphi_{lm}^2\right] \tag{8}$$

With the help of canonical conjugate momenta defined as $\pi_{lm} = \dot{\varphi}_{lm}$, we can now write down the Hamiltonian of the system. To regularize this Hamiltonian, we introduce lattice spacing a in the proper length co-ordinate as $\rho = ja$. The IR cut-off here is on the proper length, which is fixed to be $\rho_L = (N+1)a$. For each lattice point j, we obtain the corresponding lattice point in rescaled radial co-ordinate $r' = r/a$, by inverting the following expression for r_j:

$$j = \int_{\Delta_h}^{r_j} \frac{dr'}{\sqrt{f(r')}}. \tag{9}$$

where we have introduced the dimensionless parameters $\Delta_h = r_h/a$ and $r_j = r/a|_{\rho=ja}$. For convenience, we further define $f_j = f(r)|_{\rho=ja}$. It should be noted that the lattice points in radial co-ordinate $\{r_j\}$ are not equally spaced. On employing the midpoint discretization scheme[13], we obtain a fully regularized Hamiltonian:

$$H = \frac{1}{2a}\sum_{lmj}\left[\pi_{lm,j}^2 + r_{j+\frac{1}{2}}^2 f_{j+\frac{1}{2}}^{1/2}\left\{f_j^{1/4}\frac{\varphi_{lm,j}}{r_j} - f_{j+1}^{1/4}\frac{\varphi_{lm,j+1}}{r_{j+1}}\right\}^2 + f_j\left\{\Lambda^2 + \frac{l(l+1)}{r_j^2}\right\}\varphi_{lm,j}^2\right], \tag{10}$$

where $\Lambda = a^2 m_f^2$. Let us now factorize the Hamiltonian as $H = \tilde{H}/a$ and consider the following scaling transformations:

$$a \to \eta a; \quad m_f \to \eta^{-1} m_f; \quad r_h \to \eta r_h \tag{11}$$

Under these transformations, the parameters Λ and Δ_h remain invariant. The Hamiltonian H has therefore been factorized into a scale-dependent part $1/a$ and a scale-independent part $\tilde{H}$. Since the relations between entanglement measures of H and $\tilde{H}$ are well established[23], it is sufficient to work with the scale-invariant part $\tilde{H}$. It can be seen that $\tilde{H} = \sum_{lm} \tilde{H}_{lm}$ exactly resembles a network of coupled harmonic oscillators:

$$\tilde{H}_{lm} = \frac{1}{2}\left[\sum_i \pi_{lm,i}^2 + \sum_{ij} \varphi_{lm,i} K_{ij} \varphi_{lm,j}\right] \tag{12}$$

Here, K_{ij} is the coupling matrix, which contains all the relevant information about the interactions, and in which all information about entanglement entropy is encoded. To simplify further, we will focus on the massless case ($\Lambda = 0$), and also impose the Dirichlet boundary condition $\varphi_{lm,N+1} = 0$ to obtain a non-divergent scaling behavior. The coupling matrix will therefore have the following non-zero elements:

$$\begin{aligned} K_{11} =& f_1 \frac{l(l+1)}{r_1^2} + \frac{r_{3/2}^2}{r_1^2}\sqrt{f_1 f_{3/2}} \\ K_{jj\neq 1} =& f_j \frac{l(l+1)}{r_j^2} + \frac{\sqrt{f_j}}{r_j^2}\left\{r_{j+\frac{1}{2}}^2\sqrt{f_{j+\frac{1}{2}}} + r_{j-\frac{1}{2}}^2\sqrt{f_{j-\frac{1}{2}}}\right\} \\ K_{j,j+1} =& K_{j+1,j} = -\frac{r_{j+\frac{1}{2}}^2}{r_j r_{j+1}}\left\{f_{j+\frac{1}{2}}^2 f_j f_{j+1}\right\}^{1/4} \end{aligned} \tag{13}$$

Having fully described the model, we now introduce quantities associated with entanglement mechanics, namely, the entanglement entropy and entanglement energy, of a quantum field in a 4-D spherically symmetric space-time with at least one horizon[23,28]:

$$S_{\text{ent}} = -\operatorname{Tr}\rho_{red}\log\rho_{red} \quad ; \quad E_{\text{ent}} = \epsilon \operatorname{Tr}\left[\rho : H_{in} :\right], \tag{14}$$

where $: H_{in} :$ is the normal-ordered Hamiltonian corresponding to the reduced subsystem, ρ is the density matrix, and ρ_{red} is the reduced density matrix obtained by tracing over the field outside the horizon. The constant prefactor ϵ in the definition of entanglement energy accounts for the fact that it is not a unique measure[23]; it may be obtained *uniquely* by comparing with physical quantities of black-holes. While entanglement entropy (S_{ent}) essentially captures change in information content in the presence of the horizon, we may define entanglement energy (E_{ent}) as the disturbed vacuum energy or the *correlation energy* in the presence of the horizon. We further invoke the Komar relation(1) to define entanglement temperature as $T_{ent} = E_{ent}/2S_{ent}$[23]. Having formulated the quantities that describe entanglement mechanics for a given field, we proceed to simulate them in various space-times to infer its fundamental structure.

3. Entanglement mechanics of the field near space-time horizon

In this section, we discuss results from numerical simulations of entanglement mechanics in a variety of static, spherically symmetric space-times with one or more horizon(s).

3.1. *Schwarzschild Black Hole*

In the Schwarzschild space-time, $f(r) = 1 - r_h/r$, and the proper length (4) takes the form:

$$\rho = r\sqrt{1-\frac{r_h}{r}} + \frac{r_h}{2}\ln\left[\frac{r}{r_h}\left\{1+\sqrt{1-\frac{r_h}{r}}\right\}^2\right]. \tag{15}$$

where the horizon radius is $r_h = 2M$. On discretizing $\rho = ja$, we get a scale-invariant expression that connects lattice-points in the proper length and radial co-ordinates as follows:

$$j = r_j\sqrt{1-\frac{\Delta_h}{r_j}} + \frac{\Delta_h}{2}\ln\left[\frac{r_j}{\Delta_h}\left\{1+\sqrt{1-\frac{\Delta_h}{r_j}}\right\}^2\right], \tag{16}$$

where $\Delta_h = 2M/a$ and $r_j = r/a$ are dimensionless. We also see that $f_j = 1 - \Delta_h/r_j$. This confirms that the Hamiltonian in (10) is characterized by dimensionless parameters Λ and Δ_h, and is therefore invariant under the transformations:

$$a \to \eta a; \quad m_f \to \eta^{-1} m_f; \quad M \to \eta M \tag{17}$$

Now we focus on the scale-invariant Hamiltonian $\tilde{H}$, wherein we vary the rescaled horizon Δ_h, and assume that the entanglement mechanics of the field at the horizon can be approximated by tracing out the closest oscillator near the horizon. This approximation is reasonable for large values of Δ_h, wherein the radial distance of the closest oscillator from horizon is negligible ($r_1 \sim \Delta_h$).

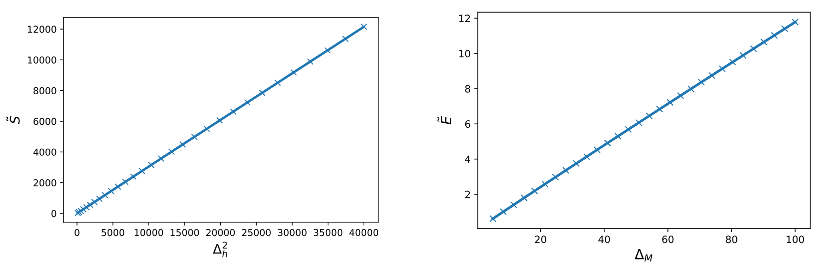
From Fig 2, we observe the following scaling relations:

$$\tilde{S} = c_s \Delta_h^2; \quad \tilde{E} = c_e \Delta_M, \tag{18}$$

where a linear fit fixes the values $c_s \sim 0.3$ and $c_e \sim 0.06$. For the original Hamiltonian H, we then have:

$$S = c_s \frac{r_h^2}{a^2}; \quad E = c_e \frac{M}{a^2}; \quad T = \frac{c_e}{4 c_s r_h} = \frac{\pi c_e}{c_s} T_H \tag{19}$$

where T_H is the Hawking temperature. From the above relations, we see that (i) entanglement entropy exhibits an area law whereas entanglement energy scales linearly with horizon radius—the latter is fundamentally different from an area-law scaling observed in Minkowski space-time, (ii) entanglement temperature is independent of the UV cut-off a, and (iii) The entanglement mechanics follows the same laws of black-hole mechanics[10]. In the following sections, we will see if these observations extend for other space-times as well.



(a) (b)

Fig. 2. Entanglement Mechanics for Schwarzschild Black Hole.

3.2. *Reissner-Nordström*

We will now probe entanglement mechanics of the field in an asymptotically flat space-time with multiple horizons. The line-element for Reissner-Nordström black hole is given by (4), where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}\,.$$

For $Q < M$, the roots are given by $r_\pm = M \pm \sqrt{M^2 - Q^2}$ where r_+ corresponds to the event-horizon and r_- refers to the internal Cauchy horizon. Thus, $f(r)$ is positive in two regions: 1. $0 < r < r_-$ and 2. $r_+ < r < \infty$.

3.2.1. *Cauchy horizon*

In terms of the dimensionless variable (χ), the Cauchy horizon is

$$r_- = Q\{\chi - \sqrt{\chi^2 - 1}\} \quad \text{where} \quad \chi = M/Q \in (1, \infty)\,.$$

To ensure that the proper length is positive definite quantity, we reverse the limits of integration in Eq. (4), i. e.,

$$\rho = \int_r^{r_h} \frac{dr}{\sqrt{f(r)}} = -\sqrt{Q^2 + r(r - 2\chi Q)} + \chi Q \ln\left[\frac{Q\sqrt{\chi^2 - 1}}{\chi Q - 2\left\{r + \sqrt{Q^2 + r(r - 2\chi Q)}\right\}}\right] \tag{20}$$

On discretizing $\rho = ja$, we convert the above expression into a dimensionless form:

$$j = -\sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)} + \chi\Delta_Q \ln\left[\frac{\Delta_Q\sqrt{\chi^2 - 1}}{\chi\Delta_Q - 2\left\{r_j + \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)}\right\}}\right], \tag{21}$$

where $\Delta_Q \equiv Q/a$ and $r_j \equiv r/a$ are both dimensionless. We also see that

$$f_j = 1 - \frac{2\chi\Delta_Q}{r_j} + \frac{\Delta_Q^2}{r_j^2}. \tag{22}$$

The resulting Hamiltonian H is factorized into a scale-dependent part $1/a$ and a scale-independent part $\tilde{H}$. The latter is completely characterized by dimensionless parameters Λ, Δ_Q and χ, all of which are invariant under the scaling transformations:

$$a \to \eta a; \quad m_f \to \eta^{-1} m_f; \quad M \to \eta M; \quad Q \to \eta Q \tag{23}$$

The IR cut-off on proper length is fixed at $r = 0$, leading to a certain discretization relation:

$$\Delta_Q\left[\chi \ln\sqrt{\frac{\chi + 1}{\chi - 1}} - 1\right] = N + 1 \tag{24}$$

The above relation tells us that if we fix Δ_q, then χ is discretized and vice versa. Here, we will consider the scenario where horizon changes on account of varying Δ_Q while keeping χ fixed. Physically, this corresponds to varying both mass and charge of the black hole proportionately to account for particles with a fixed mass-charge ratio (χ) that are entering the event horizon. As a result, both mass and charge have equally spaced discrete spectra:

$$Q_N = \frac{(N+1)a}{\left(\chi \ln \sqrt{\frac{\chi+1}{\chi-1}} - 1\right)}; \quad M_N = \chi Q_N; \tag{25}$$

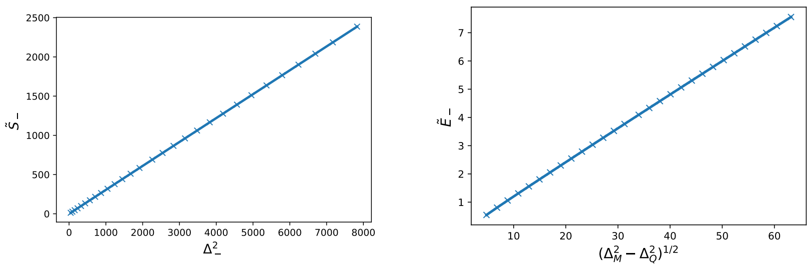


Fig. 3. Entanglement Mechanics at RN Cauchy horizon, when $\chi = 1.1$.

3.2.2. *Event horizon*

In terms of the dimensionless variable (χ), the event horizon is

$$r_+ = Q(\chi + \sqrt{\chi^2 - 1}) .$$

From Eq. (4), we obtain:

$$\rho = \sqrt{Q^2 + r(r - 2\chi Q)} + \chi Q \ln \left[\frac{r - \chi Q + \sqrt{Q^2 + r(r - 2\chi Q)}}{Q\sqrt{\chi^2 - 1}} \right] . \tag{26}$$

On discretizing $\rho = ja$, we convert the above expression into a dimensionless form:

$$j = \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)} + \chi\Delta_Q \ln \left[\frac{r_j - \chi\Delta_Q + \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)}}{\Delta_Q\sqrt{\chi^2 - 1}} \right] \tag{27}$$

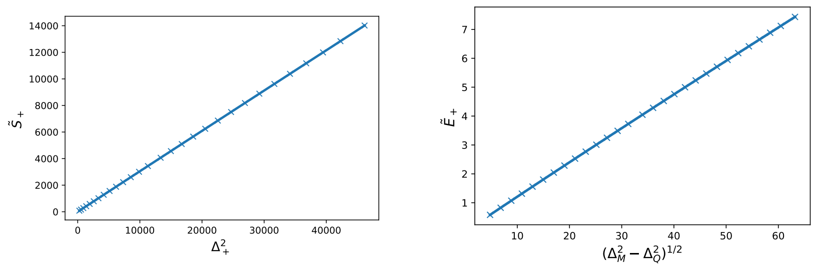
where $\Delta_Q \equiv Q/a$ and $r_j \equiv r/a$ are both dimensionless. From Figs 3 and 4, we obtain the following scaling relations for scale-invariant system $\tilde{H}$:

$$\tilde{S}_\pm = c_s \Delta_\pm^2; \quad \tilde{E}_\pm = c_e \sqrt{\Delta_M^2 - \Delta_Q^2}, \tag{28}$$

where a linear fit fixes the values $c_s \sim 0.3$ and $c_e \sim 0.12$ for both horizons. It can also be seen from here that in the limit $\Delta_Q \to 0$, we recover the values of c_e and c_s for Schwarzschild (18). As discussed above, $\tilde{S}_-$ and $\tilde{E}_-$ have discrete spectra. Since the entanglement energy is identical for both the horizons, we may therefore write for the total Hamiltonian (H):

$$S_\pm = c_s \frac{r_\pm^2}{a^2}; \quad E_+ = c_e \frac{\sqrt{M^2 - Q^2}}{a^2}; \quad T^{(\pm)} = \frac{\pi c_e}{c_s} T_H^{(\pm)} \tag{29}$$

where $T_H^{(-)}$ and $T_H^{(+)}$ are the Hawking temperatures of Cauchy and event horizon, respectively.



(a) (b)

Fig. 4. Entanglement Mechanics at RN event horizon, when $\chi = 1.1$.

3.3. *Schwarzschild de-Sitter*

We now move on to a space-time with multiple horizons that is asymptotically non-flat. A Schwarzschild black hole (of mass M) in a de-Sitter space-time (of radius l) is described by:

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{l^2},$$

This space-time also has two horizons — r_b (event horizon) and r_c (cosmological horizon)[29]:

$$r_- = -\frac{2l}{\sqrt{3}} \cos\frac{\theta}{3}; \quad r_b = \frac{2l}{\sqrt{3}} \cos\frac{\pi + \theta}{3}; \quad r_c = \frac{2l}{\sqrt{3}} \cos\frac{\pi - \theta}{3} \tag{30}$$

where r_- is the third negative root, $\theta = \cos^{-1}(3\sqrt{3}\chi)$ and $\chi = M/l \in \left[0, 1/(3\sqrt{3})\right]$. $f(r)$ is positive in the region between the two horizons. Hence, in this region, we have two definitions for proper length — one w.r.t. the event horizon r_b which we refer to as ρ_b and the second w.r.t. the cosmological horizon r_c which we refer to as ρ_c.

3.3.1. *Event Horizon*

The proper length with respect to the event horizon r_b is obtained as follows[23]:

$$\rho_b = \frac{2r_b l}{\sqrt{r_c(r_b - r_-)}}\Pi\left(\vartheta, \alpha^2, k\right), \tag{31}$$

where,

$$\vartheta = \sin^{-1}\sqrt{\frac{r_c(r-r_b)}{r(r_c-r_b)}}; \quad \alpha^2 = 1 - \frac{r_b}{r_c}; \quad k^2 = \frac{r_-(r_b - r_c)}{r_c(r_b - r_-)}. \tag{32}$$

On discretizing proper length $\rho_b = ja$, we convert the above expression into a dimensionless form:

$$j = \frac{2\Delta_b \Delta_l}{\sqrt{\Delta_c(\Delta_b - \Delta_-)}}\Pi\left(\vartheta, \alpha^2, k\right). \tag{33}$$

In terms of the dimensionless variables $\Delta_l = l/a$ and $r_j = r/a$, we have

$$f_j = 1 - \frac{2\Delta_M}{r_j} - \frac{r_j^2}{\Delta_l^2}, \tag{34}$$

and,

$$\Delta_- = -\frac{2\Delta_l}{\sqrt{3}}\cos\frac{\theta}{3}; \quad \Delta_b = \frac{2\Delta_l}{\sqrt{3}}\cos\frac{\pi+\theta}{3}; \quad \Delta_c = \frac{2\Delta_l}{\sqrt{3}}\cos\frac{\pi-\theta}{3};$$

$$\vartheta = \sin^{-1}\sqrt{\frac{\Delta_c(r_j - \Delta_b)}{r_j(\Delta_c - \Delta_b)}}; \quad \alpha^2 = 1 - \frac{\Delta_b}{\Delta_c}; \quad k^2 = \frac{\Delta_-(\Delta_b - \Delta_c)}{\Delta_c(\Delta_b - \Delta_-)}. \tag{35}$$

As a result, the Hamiltonian $\tilde{H}$ is fully characterized by dimensionless parameters Δ_l and Δ_M, both of which are invariant under the scaling transformations:

$$a \to \eta a; \quad m_f \to \eta^{-1} m_f; \quad M \to \eta M; \quad l \to \eta l \tag{36}$$

In the case of SdS, the IR cut-off on proper length is automatically fixed as we restrict ourselves to the region $\tilde{r}_b \le r_j \le \tilde{r}_c$:

$$N + 1 = \frac{2\tilde{r}_b \Delta_l}{\sqrt{\tilde{r}_c(\tilde{r}_b - \tilde{r}_-)}}\Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \tag{37}$$

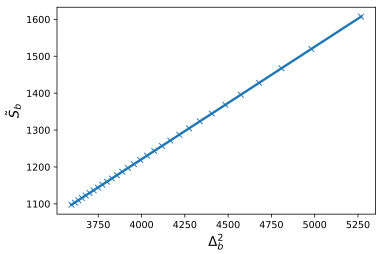
This is a discretization relation similar to what was obtained for RNBH. We will consider the case where we fix Δ_M and vary Δ_l by varying N. This is to ensure that χ is always between $[0, 1/(3\sqrt{3}]$. From Fig 5, we obtain the following scaling relations for the scale-invariant Hamiltonian ($\tilde{H}$):

$$\tilde{S}_b \sim c_s \Delta_b^2; \qquad \tilde{E}_b \sim c_e(3\Delta_M - \Delta_b) \tag{38}$$

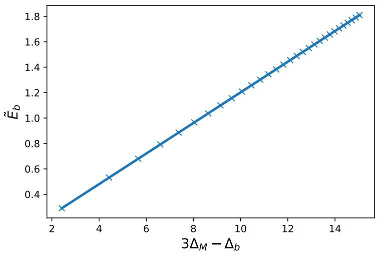
where, $c_s \sim 0.3$ and $c_e \sim 0.12$ are the best-fit numerical values. In the limit $\Delta_l \to \infty$, we recover the prefactors of the Schwarzschild black hole (18). For the total Hamiltonian (H), the scaling relations become:

$$S_{b,N} \sim c_s \frac{r_{b,N}^2}{a^2}; \qquad E_{b,N} \sim c_e \frac{3M - r_{b,N}}{a^2}; \qquad T^{(b)} = \frac{\pi c_e}{c_s} T_H^{(b)} \tag{39}$$

where $T_H^{(b)}$ is the Hawking temperature of the event horizon in SdS[29,30].



(a)



(b)

Fig. 5. Entanglement Mechanics at SdS event horizon, when $\Delta_M = 25$.

3.3.2. *Cosmological Horizon*

To explore the scaling properties of cosmological horizon, we define proper distance r_c as follows[31]:

$$\rho_c = \frac{2l}{\sqrt{r_c(r_b - r_-)}} \left[r_- F(\vartheta, k) - (r_c - r_-)\, \Pi\left(\vartheta, \alpha^2, k\right)\right], \tag{40}$$

where,

$$\sin\vartheta = \sqrt{\frac{(r_b - r_-)(r_c - r)}{(r_c - r_b)(r - r_-)}}; \quad \alpha^2 = \frac{r_b - r_c}{r_b - r_-}, \tag{41}$$

and the definition of k is the same as Eq. (32). On discretizing proper length $\rho_c = ja$, we convert the above expression into a dimensionless form:

$$j = \frac{2\Delta_l}{\sqrt{\Delta_c(\Delta_b - \Delta_-)}} \left[\Delta_- F(\vartheta, k) - (\Delta_c - \Delta_-)\, \Pi\left(\vartheta, \alpha^2, k\right)\right], \tag{42}$$

where in terms of dimensionless variables $\Delta_l = l/a$ and $r_j = r/a$, we can also rewrite:

$$\sin\vartheta = \sqrt{\frac{(\Delta_b - \Delta_-)(\Delta_c - r_j)}{(\Delta_c - \Delta_b)(r_j - \Delta_-)}}; \quad \alpha^2 = \frac{\Delta_b - \Delta_c}{\Delta_b - \Delta_-}, \tag{43}$$

Except for ϑ and α given above, the parameters used here follow the same definition as in (35). Now we impose an IR cut-off on proper length to restrict ourselves in the region $\tilde{r}_b \leq r_j \leq \tilde{r}_c$:

$$N + 1 = \frac{2\Delta_l}{\sqrt{\Delta_c(\Delta_b - \Delta_-)}} \left[\Delta_- K(k) - (\Delta_c - \Delta_-)\, \Pi\left(\frac{\pi}{2}, \alpha^2, k\right)\right] \tag{44}$$

This expression relates the number of oscillators N, Δ_l and χ, and we only need to fix two of these to fix the third. We will fix Δ_M here as we did for the event horizon, which leaves Δ_l with a discrete spectrum. From Fig 6, we see that the spectra for Δ_l obtained from ρ_b and ρ_c coincide exactly, and therefore the two discretization relations (37) and (44) are identical.

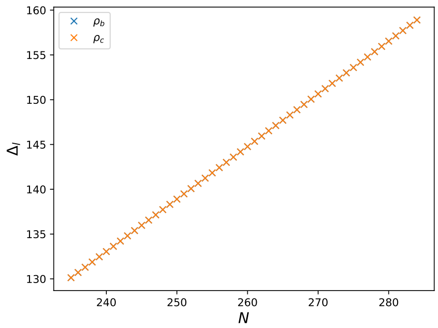


Fig. 6. Discretization of Δ_l from the cutoffs on proper lengths ρ_b and ρ_c, for $\Delta_M = 25$.

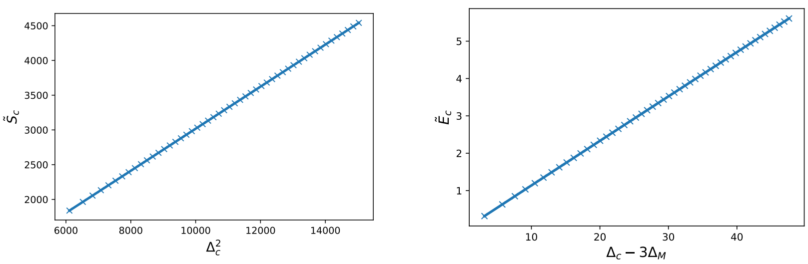
From Fig 7, we obtain the following scaling relations for the scale-invariant system $\tilde{H}$:

$$\tilde{S}_c \sim c_s \Delta_c^2; \qquad \tilde{E}_c \sim c_e \left[\Delta_c - 3\Delta_M\right] \tag{45}$$

where $c_s \sim 0.3$ and $c_e \sim 0.12$ are the best-fit numerical values. For the total Hamiltonian H, the scaling relations become:

$$S_{c,N} \sim c_s \frac{r_{c,N}^2}{a^2}; \quad E_{c,N} \sim c_e \frac{r_{c,N} - 3M}{a^2}; \quad T^{(c)} \sim \frac{\pi c_e}{c_s} T_H^{(c)}, \tag{46}$$

where $T_H^{(c)}$ is the Hawking temperature of the cosmological horizon in SdS.



(a) (b)

Fig. 7. Entanglement Mechanics at SdS cosmological horizon, when $\Delta_M = 25$.

4. Black hole thermodynamics from entanglement mechanics

On numerically simulating the quantum entanglement mechanics near the horizon r_h for a variety of such space-times, we observe some universal properties associated with their scaling relations. Before we proceed, we first invoke the definition of entanglement energy wherein the pre-factor ϵ was introduced. Here, we may fix this pre-factor as $\epsilon \sim 1.26$ upon imposing the condition that the entanglement temperature is identical to Hawking temperature ($T_{ent} = T_H$). While this rescales the proportionality constant c_e obtained numerically, the simulations nevertheless capture a universal one-to-one correspondence[23] between entanglement mechanics and black hole thermodynamics:

$$E_{\rm ent} = \frac{c_e}{a^2} E_{\rm Komar}; \quad S_{\rm ent} = \frac{c_s}{\pi a^2} S_{\rm BH}; \quad T_{\rm ent} = T_H \,. \tag{47}$$

Note that $E_{\rm ent}$ and $S_{\rm ent}$ depend on the UV cut-off (a), where as the temperature does not. Moreover, even with the pre-factor ϵ being assigned a new value, the constants of proportionality are found to be universal across all black-hole space-times, irrespective of whether they are asymptotically flat or non-flat:

$$c_e \sim 0.0955; \quad c_s \sim 0.3 \,. \tag{48}$$

Let us put these results in perspective: We have related quantum scalar fields near the horizon with the thermodynamic observables that describe any black-hole space-time (cf. Table 1). The fact that this is true for all 4-D spherically symmetric space-time suggests that entanglement of the quantum fields near the horizon carries crucial information about the black-hole thermodynamics. The above relations (47) go further and lead to the following universal and cut-off independent (a) result:

$$E_{\rm ent} = 2T_{\rm ent}\, S_{\rm ent} \qquad \Longleftrightarrow \qquad E_{\rm Komar} = 2T_H\, S_{\rm BH} \tag{49}$$

It is important to note that the above relation does not imply equality of their respective counterparts. On rearranging this relation further, we obtain the generalized Smarr formula of black hole thermodynamics, as summarized in Table 1.

Space-time	Entanglement Structure	Thermodynamic Structure	Smarr formula	Pressure	Potential
Schwarzschild	$S_{\rm ent} = (c_s/a^2)r_h^2$ $E_{\rm ent} = (c_e/a^2)M$	$S_{BH} = \pi r_h^2$ $E_{\rm Komar} = M$	$M = 2T_{\rm H}\, S_{BH}$	—	—
Reissner-Nordström	$S_+ = (c_s/a^2)r_+^2$ $E_+ = (c_e/a^2)\sqrt{M^2 - Q^2}$	$S_{BH} = \pi r_+^2$ $E_{Komar} = \sqrt{M^2 - Q^2}$	$M = 2T_{\rm H}\, S_{BH} + Q^2/r_+$	—	Q/r_+
Schwarzschild-AdS	$S_{\rm ent} = (c_s/a^2)r_h^2$ $E_{\rm ent} = (c_e/a^2)[3M - r_h^2]$	$S_{BH} = \pi r_h^2$ $E_{Komar} = 3M - r_h^2$	$M = 2T_{\rm H}\, S_{BH} - r_h^3/l^2$	$3/8\pi l^2$	—
Schwarzschild-dS	$S_b = (c_s/a^2)r_b^2$ $E_b = (c_e/a^2)[3M - r_b^2]$	$S_{BH} = \pi r_b^2$ $E_{Komar} = 3M - r_b^2$	$M = 2T_{\rm H}\, S_{BH} + r_b^3/l^2$	$-3/8\pi l^2$	—

5. Conclusion

The Smarr formula for asymptotically flat and non-flat space-times is generally derived using Komar integral relations[32] or via the first law of thermodynamics with the help of scaling relations[33], both of which makes use of Killing potential. In this talk, we developed an independent approach towards the derivation of the generalized Smarr formula, which essentially describes the equation of state for black holes. The intrinsically quantum phenomenon of entanglement pertaining to the field near a horizon, gives rise to not just the thermodynamic quantities associated with the space-time, but also the exact relation connecting the same. This in turn gives entanglement mechanics an upper hand in addressing the universality problem of black hole entropy. While a wealth of other approaches may explain black-hole entropy, entanglement mechanics *alone* expands the analogy to include other thermodynamic quantities as well.

The results further complements two earlier results: First, Jacobson argued that entanglement provides a link between the presence of matter and the space-time geometry[34]. Second, in quantum gravity, Perez conjectured that degrees of freedom hidden from the classical space-time description but correlated to matter fields are necessary to maintain unitarity in the global evolution and prevent the information loss[35]. Our analysis here provides a crucial link to these results, placing entanglement at the center of a fundamental connection between space-time and matter, thereby holding the key to understanding black-hole horizons.

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