Black Hole Thermodynamics from Entanglement Mechanics

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Outline

1. Motivation
2. Entanglement Mechanics : Formalism
3. Black hole thermodynamics from quantum entanglement
4. Conclusions and Future Directions

Basics of Entanglement

- Quantum Entanglement : $|\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle$. Iff the state is entangled:

$$\langle\Psi|\mathscr{O}_1 \otimes \mathscr{O}_2|\Psi\rangle \neq \langle\Psi|\mathscr{O}_1|\Psi\rangle\langle\Psi|\mathscr{O}_2|\Psi\rangle \tag{1}$$

- The two-point function of a scalar field: [*Unruh & Wald '17*]

$$\langle\Psi\,|\phi\,(x_1)\,\phi\,(x_2)|\,\Psi\rangle \propto \frac{1}{\sigma\,(x_1, x_2)} \tag{2}$$

where $\sigma(x_1, x_2)$ is the squared geodesic distance.

Universality Problem of Black Hole Entropy

Area-law of GS entanglement entropy from field theory:

$$S \sim c_0 \frac{A}{a^2} \quad \Leftrightarrow \quad S_{BH} = \frac{A}{4}$$

BH entropy can be sufficiently explained by:

- Noether Charge [Wald '93]
- D-Branes [Strominger & Vafa '96]
- Spin-networks [Ashtekar et al '96]
- Conformal Symmetry [Carlip '99]
- Entanglement Entropy [Bombelli et al '86, Das & SS '08]

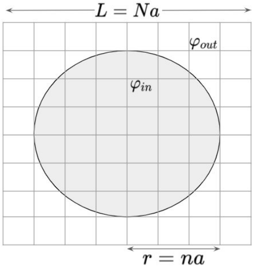
How can we break this degeneracy? ("Universality Problem") [Carlip '07]

Scalar Field in Spherically Symmetric Space-times

Hamiltonian for a massless scalar field:

$$H = \frac{1}{2} \int d^3x \left[\pi^2 + (\nabla\varphi)^2\right]$$

Partial wave expansion gives us an effective (1 + 1)-D system.



We discretize the field as a collection of oscillators at $r = ja$:

$$H = \frac{1}{2a} \sum_{lmj} \left[\pi^2_{lm,j} + \frac{l(l+1)}{j^2}\varphi^2_{lm,j} + \left(j + \frac{1}{2}\right)^2 \left\{\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right\}^2\right] = \frac{1}{a}\sum_{lm} H_{lm}. \quad (3)$$

Entanglement Entropy

The Hamiltonian can be bipartited as follows:

$$H_{lm} = \frac{1}{2}\left[\sum_i \pi^2_{lm,i} + \sum_{ij} K_{ij}\varphi_{lm,i}\varphi_{lm,j}\right] = \begin{bmatrix} H_{in} & H_{int} \\ H_{int} & H_{out} \end{bmatrix}. \tag{4}$$

We focus on the ground state Ψ_0. To obtain reduced density matrix, we trace out the first n oscillators:

$$\rho_{red}(\{\varphi_{lm,\alpha}\},\{\varphi'_{lm,\beta}\}) = \int \prod_{a=1}^{n} d\varphi_{lm,a}\Psi_0^*(\{\varphi_{lm,b}\},\{\varphi_{lm,\alpha}\})\Psi_0(\{\varphi_{lm,c}\},\{\varphi'_{lm,\beta}\}) \tag{5}$$

Let $\{p_i\}$ be the eigenvalues of ρ_{red}. The von-Neumann entropy of the system is:

$$S_{lm} = -\sum_i p_i \ln p_i; \qquad S = \sum_l (2l+1) S_{lm} \tag{6}$$

Entanglement Energy

We define "entanglement energy" as follows: [Mukohyama et al '98, SMC & SS '20]

$$E = \epsilon \int \prod_{A=1}^{N} d\bar{\varphi}_{lm,A} \left\langle \{\bar{\varphi}_{lm,B}\} \right| : H_{in} : \rho \left| \{\bar{\varphi}_{lm,C}\} \right\rangle , \tag{7}$$

where $\omega^{in} = K_{in}^{1/2}$ and the normal ordered Hamiltonian H_{in} is given by:

$$\begin{aligned} : H_{in} : &= -\frac{1}{2}\delta^{ab}\left(\frac{\partial}{\partial \varphi^{a}_{lm}} - \omega^{in}_{ac}\varphi^{c}_{lm}\right)\left(\frac{\partial}{\partial \varphi^{b}_{lm}} + \omega^{in}_{bd}\varphi^{d}_{lm}\right) \\ &= -\frac{1}{2}U^{ab}\left(\frac{\partial}{\partial \bar{\varphi}^{A}_{lm}} - \bar{\omega}^{in}_{AC}\bar{\varphi}^{C}_{lm}\right)\left(\frac{\partial}{\partial \bar{\varphi}^{B}_{lm}} + \bar{\omega}^{in}_{BD}\bar{\varphi}^{D}_{lm}\right) . \end{aligned} \tag{8}$$

For now, we set $\epsilon = 1$.

Schwarzschild Black Hole

$$\Delta_M = M/a$$

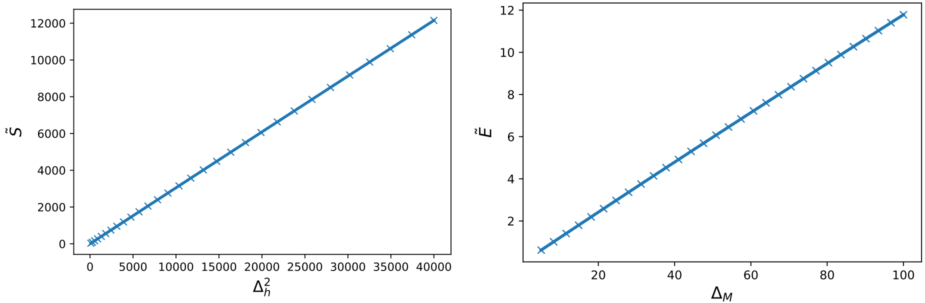
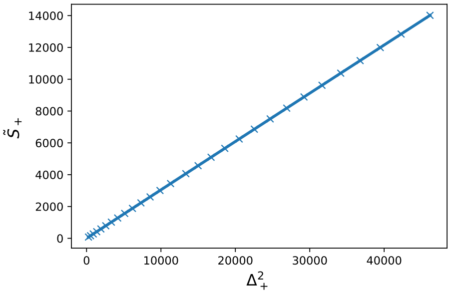


Figure: Entanglement Mechanics for Schwarzschild Black Hole.

$$S = c_s \frac{r_h^2}{a^2}; \quad E = c_e \frac{M}{a^2}; \quad T = \frac{c_e}{4 c_s r_h} = \frac{\pi c_e}{c_s} T_H$$

Reissner-Nordström Event Horizon

$$\Delta_M = M/a; \quad \Delta_Q = Q/a; \quad \chi = M/Q; \quad \Delta_+ = r_+/a$$



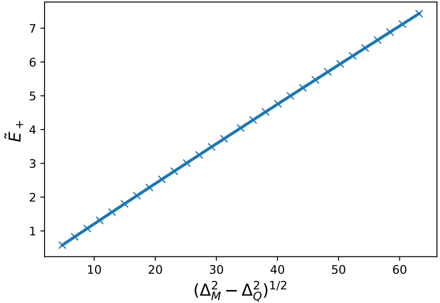
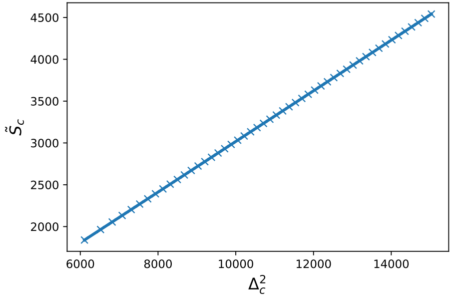


Figure: Entanglement Mechanics at RN event horizon, when $\chi = 1.1$.

$$S_+ = c_s \frac{r_+^2}{a^2}; \quad E_+ = c_e \frac{\sqrt{M^2 - Q^2}}{a^2}; \quad T^{(+)} = \frac{\pi c_e}{c_s} T_H^{(+)}$$

SdS Cosmological Horizon

$$\Delta_M = M/a; \quad \Delta_l = l/a; \quad \Delta_c = r_c/a$$



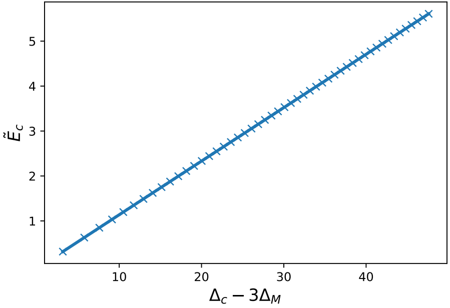


Figure: Entanglement Mechanics at SdS cosmological horizon, when $\Delta_M = 25$.

$$S_{c,N} \sim c_s \frac{r_{c,N}^2}{a^2}; \quad E_{c,N} \sim c_e \frac{r_{c,N} - 3M}{a^2}; \quad T^{(c)} \sim \frac{\pi c_e}{c_s} T_H^{(c)}$$

One-to-One Correspondence

On studying the entanglement mechanics for Schwarzschild, de Sitter, RNBH, Schwarzschild-AdS and Schwarzschild-dS space-times:

- We observe a one-to-one correspondence as follows:

$$E = \frac{c_e}{a^2} E_{Komar}; \quad S = \frac{c_s}{\pi a^2} S_{BH}; \quad T = \frac{\pi c_e}{c_s} T_H \tag{9}$$

- The following relations are universal:

$$c_e \sim 0.12; \quad c_s \sim 0.3; \quad \frac{T}{T_H} = \frac{\pi c_e}{c_s} \sim 1.26 \tag{10}$$

- While the constants of proportionality for E and S depend on UV cut-off a, entanglement temperature T is cut-off independent.

Fixing ϵ

- We now impose the physical condition $T = T_H$. This fixes $\epsilon \sim 1.26$. As a result:

$$c_e \to \frac{c_e}{1.26}; \quad \frac{\pi c_e}{c_s} \to 1; \tag{11}$$

- The following relations are universal and cut-off independent:

$$E = 2T_H S \quad \Leftrightarrow \quad E_{Komar} = 2T_H S_{BH} \tag{12}$$

[*Padmanabhan '04, '05*]

- From the modified scaling relations, which satisfy $E = 2TS$, we may derive the Smarr formula of black-hole thermodynamics.

Smarr-Formula from Entanglement Scaling Relations

Space-time	Entanglement Structure	Thermodynamic Structure	Smarr formula	Pressure	Potential
Schwarzschild	$S = (c_s/a^2)r_h^2$ $E = (c_e/a^2)M$	$S_{BH} = \pi r_h^2$ $E_{Komar} = M$	$M = 2TS_{BH}$	—	—
Reissner-Nordström	$S_+ = (c_s/a^2)r_+^2$ $E_+ = (c_e/a^2)\sqrt{M^2 - Q^2}$	$S_{BH} = \pi r_+^2$ $E_{Komar} = \sqrt{M^2 - Q^2}$	$M = 2TS_{BH} + Q^2/r_+$	—	Q/r_+
Schwarzschild-AdS	$S = (c_s/a^2)r_h^2$ $E = (c_e/a^2)[3M - r_h^2]$	$S_{BH} = \pi r_h^2$ $E_{Komar} = 3M - r_h^2$	$M = 2TS_{BH} - r_h^3/l^2$	$3/8\pi l^2$	—
Schwarzschild-dS	$S_b = (c_s/a^2)r_b^2$ $E_b = (c_e/a^2)[3M - r_b^2]$	$S_{BH} = \pi r_b^2$ $E_{Komar} = 3M - r_b^2$	$M = 2TS_{BH} + r_b^3/l^2$	$-3/8\pi l^2$	—

Table: Summary of entanglement mechanics and event-horizon thermodynamics. Since we have set $\epsilon \sim 1.26$, they also satisfy $T = T_H$ and $E = 2TS$ universally.

Conclusions and Future Directions

- One-to-one correspondence does not imply equality. Yet, the Smarr formula structure of black-hole thermodynamics is exactly recovered from entanglement. This offers a possible resolution to the universality problem.
- Completely new and independent derivation of the generalized Smarr formula. Does this hold in higher-dimensions/modified theories?
- Entanglement mechanics do not explicitly use Einstein's equations. Instead, it is the scalar field φ that captures information about the dynamical properties of background space-time!
- Can entanglement mechanics throw light on Noether charge? [*Fursaev & Frolov '98*]

Thank You!

Spherically Symmetric Space-times with Horizon

Consider the following line element:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2. \tag{13}$$

In terms of proper length ρ:

$$ds^2 = -f(r)dt^2 + d\rho^2 + r^2 d\Omega^2; \qquad \rho = \int_{r_h}^{r} \frac{dr}{\sqrt{f(r)}}. \tag{14}$$

On discretizing $\rho = ja$ and hence $r(\rho) = r_j a$:

$$H = \frac{1}{2a} \sum_{lmj} \left[\pi_{lm,j}^2 + r_{j+\frac{1}{2}}^2 f_{j+\frac{1}{2}}^{1/2} \left\{ f_j^{1/4} \frac{\varphi_{lm,j}}{r_j} - f_{j+1}^{1/4} \frac{\varphi_{lm,j+1}}{r_{j+1}} \right\}^2 + \frac{l(l+1)}{r_j^2} f_j \varphi_{lm,j}^2 \right]. \tag{15}$$

Schwarzschild Black Hole

Can be described by the following line element and proper length:

$$f(r) = 1 - 2M/r \quad ; \quad \rho = r\sqrt{1-\frac{2M}{r}} + M \ln\left[\frac{r}{2M}\left\{1+\sqrt{1-\frac{2M}{r}}\right\}^2\right] \tag{16}$$

On discretizing $\rho = ja$, we get a scale-invariant expression that connects lattice-points in the proper length and radial co-ordinates as follows:

$$j = r_j\sqrt{1-\frac{2\Delta_M}{r_j}} + \Delta_M \ln\left[\frac{r_j}{2\Delta_M}\left\{1+\sqrt{1-\frac{2\Delta_M}{r_j}}\right\}^2\right] \quad ; \quad f_j = 1 - 2\Delta_M/r_j \tag{17}$$

where $\Delta_M = M/a$ and $r_j = r/a$ are dimensionless.

Reissner-Nordström Event Horizon

Can be described by the following line element:

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{18}$$

For $Q < M$, the roots are given by $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$. Thus, $f(r)$ is positive in two regions: $0 < r < r_-$ and $r_+ < r < \infty$.

$$j = \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)} + \chi\Delta_Q \ln\left[\frac{r_j - \chi\Delta_Q + \sqrt{\Delta_Q^2 + r_j(r_j - 2\chi\Delta_Q)}}{\Delta_Q\sqrt{\chi^2 - 1}}\right] \tag{19}$$

where $\Delta_Q \equiv Q/a$, $\chi = M/Q$ and $r_j \equiv r/a$ are all dimensionless

SdS Cosmological Horizon

Can be described by the following line element:

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{l^2} \tag{20}$$

Once more, we have two horizons — r_b (event horizon) and r_c (cosmological horizon):

$$r_b = \frac{2l}{\sqrt{3}} \cos \frac{\pi + \theta}{3}; \quad r_c = \frac{2l}{\sqrt{3}} \cos \frac{\pi - \theta}{3} \tag{21}$$

where $\theta = \cos^{-1}(3\sqrt{3}\chi)$ and $\chi = M/l \in \left[0, 1/(3\sqrt{3})\right]$. $f(r)$ is positive in the region between the two horizons.